Dependence testing in arrays

- Review ZIV and SIV dependences
- MIV – GCD test
- Introduce Banerjee Inequality
- Overview of trapezoidal Banerjee

Conditions for dependence

- The same location must be referenced twice
- At least one of them must be a write
- There exists a path between the two references

ZIV and SIV testing

```
DO i = 1, N
        DO j = 1, N
                A[i] = B[N]
                B[1] = A[j]
        ENDDO
ENDDO
```

How do you deal with MIV?

```
DO i = 1, N
        DO j = 1, N
                A[i + 4 * j, 1, i] = B[N]
                B[1] = A[2 * j - i + N, 1, i]
        ENDDO
ENDDO
```

General MIV

A[$f(x_1, x_2, ..., x_n)$] $= \ldots$
$\ldots =$ A[$g(y_1, y_2, ..., y_n)$]

Equation 3.13

$$h(x_1, x_2, ..., x_n, y_1, y_2, ..., y_n) =$$
$$f(x_1, x_2, ..., x_n) - g(y_1, y_2, ..., y_n) = 0$$

Theorem 3.2

Eq 3.13 has a real solution over region R iff

$$min_R h \leq 0 \leq max_R h$$

Equation 3.17

$$a_0 - b_0 + a_1x_1 - b_1y_1 + ... + a_nx_n - b_ny_n = 0$$

Equation 3.18

$$a_1x_1 - b_1y_1 + ... + a_nx_n - b_ny_n = b_0 - a_0$$

Theorem 3.3

GCD Test : Eq 3.18 has a solution iff $gcd(a_1, ..., a_n, b_1, ..., b_n)$ divides $b_0 - a_0$

But often $gcd(a_1, ..., a_n, b_1, ..., b_n) = 1$ which divides everything !

We require a general dependence test for MIV

The Banerjee Inequality

Why is it better than GCD?

— It considers the limits of region R

Definition 3.1

Let $h_i^+ = max_{R_i} h_i(x_i, y_i)$ and $h_i^- = min_{R_i} h_i(x_i, y_i)$, where

$$h_i(x_i, y_i) = a_i x_i - b_i y_i$$

the region R_i is defined by

$$L_i \leq x_i, y_i \leq U_i \text{ and } x_i D_i y_i$$

where D_i is the direction vector element for the ith position

Definition 3.2

We call a^+ the positive portion of a. Likewise a^- is the negative portion of a.

```
if (a > 0)
        a+ = a
        a- = 0
else
        a+ = 0
        a- = -a
```

$$a = a^+ - a^-$$

Lemma 3.1

Let t, s, and z denote real numbers.
If $0 \leq z \leq s$, then

$$-t^{-}s \leq tz \leq t^{+}s$$

and there exist values z_1, z_2 such that

$$tz_1 = -t^{-}s \text{ and } tz_2 = t^{+}s$$

Proof of Lemma 3.1

$$-t^- s \le tz \le t^+ s$$
$$tz_1 = -t^- s \text{ and } tz_2 = t^+ s$$

Case 1: $t \ge 0$. $t^+ = t$ and $t^- = 0$.

$$0 \le tz = t^+ z \le t^+ s \text{ since } 0 \le z \le s$$

$z_1 = 0$ and $z_2 = s$

Case 2: $t < 0$. $t^+ = 0$ and $t^- = -t$.

$$-t^- s \le -t^- z = tz \le 0$$

$z_1 = s$ and $z_2 = 0$.

Lemma 3.2

Let t, l, u, and z denote real numbers. If

$$l \leq z \leq u$$

then

$$-t^{-}u + t^{+}l \leq tz \leq t^{+}u - t^{-}l$$

and there exist values z_1, z_2 such that

$$tz_1 = -t^{-}u + t^{+}l \text{ and } tz_2 = t^{+}u - t^{-}l$$

Proof of Lemma 3.2

Rearranging $l \leq z \leq u$ yields $0 \leq z - l \leq u - l$. Apply lemma 3.1

$$-t^{-}(u-l) \leq t(z-l) \leq t^{+}(u-l)$$

add tl everywhere

$$-t^{-}u + (t+t^{-})l \leq tz \leq t^{+}u - (t^{+}-t)l$$

We know that $t^{+} = t + t^{-}$ and $t^{-} = t^{+} - t$. Substitute!

The rest of the proof mirrors lemma 3.1

Definition 3.3

$$
\begin{aligned}
H_i^-(<) &= -(a_i^- + b_i)^+(U_i - 1)+ \\
&\qquad [(a_i^- + b_i)^- + a_i^+]L_i - b_i \\
H_i^+(<) &= (a_i^+ - b_i)^+(U_i - 1)- \\
&\qquad [(a_i + - - b_i)^- + a_i^-]L_i - b_i \\
H_i^-(=) &= -(a_i - b_i)^-U_i + (a_i - b_i)^+L_i \\
H_i^+(=) &= (a_i - b_i)^+U_i - (a_i - b_i)^-L_i \\
H_i^-(>) &= -(a_i - b_i^+)^-(U_i - 1)+ \\
&\qquad [(a_i - b_i^+)^+ + b_i^-]L_i + a_i \\
H_i^+(>) &= (a_i + b_i^-)^+(U_i - 1)- \\
&\qquad [(a_i + b_i^-)^- + b_i^+]L_i + a_i \\
H_i^-(*) &= -a_i^-U_i^x + a_i^+L_i^x - b_i^+U_i^y + b_i^-L_i^y \\
H_i^+(*) &= a_i^+U_i^x - a_i^-L_i^x + b_i^-U_i^y - b_i^+L_i^y
\end{aligned}
$$

Example when $H_i(*)$ is useful

```
DO i = 1, N
        DO j = 1, N/2
               A(i+j) = B(i)
        ENDDO
        DO j = N/2 + 1, N
               B(i) = A(i+j) + 5
        ENDDO
ENDDO
```

Lemma 3.3

Let $h_i(x_i, y_i) = a_i x_i - b_i y_i$ and let R_i be described by Definition 3.1.

$$L_i \leq x_i,\ y_i \leq U_i \text{ and } x_i D_i y_i$$

Then the following equations hold :

$$\min_R h = h_i^- = H_i^-(D)$$
$$\max_R h = h_i^+ = H_i^+(D)$$

Case 1: D = "=". $L_i \leq x_i = y_i \leq U_i$.
Substituting x_i for y_i produces ...

$$L_i \leq x_i \leq U_i$$
$$h_i = a_i x_i - b_i y_i = (a_i - b_i) x_i$$

Using Lemma 3.2 we get

$$-t^{-}U_i + t^{+}L_i \leq t x_i \leq t^{+}U_i - t^{-}L_i$$

Replacing t with $(a_i - b_i)$ we get

$$-(a_i - b_i)^{-}U_i + (a_i - b_i)^{+}L_i \leq h_i \leq (a_i - b_i)^{+}U_i - (a_i - b_i)^{-}L_i$$

This is exactly what was predicted

Case 2 : D = "<". $L_i \leq x_i < y_i \leq U_i$. Since x_i and y_i are integers we can do the following

$$L_i \leq x_i \leq y_i - 1 \leq U_i - 1$$

$$h_i = a_i x_i - b_i y_i = a_i x_i - b_i(y_i - 1) - b_i$$

Apply Lemma 3.2 to $L_i \leq x_i \leq y_i - 1$ and replace t with a_i

$$-a_i^-(y_i - 1) + a_i^+ L_i \leq a_i x_i \leq a_i^+(y_i - 1) - a_i^- L_i$$

If we add $(-b_i(y_i - 1) - b_i)$ to all sides we get

$$-a_i^-(y_i - 1) + a_i^+ L_i - b_i(y_i - 1) - b_i \leq h_i \leq a_i^+(y_i - 1) - a_i^- L_i - b_i(y_i - 1) - b_i$$

We can factor out $(y_i - 1)$

$$(y_i - 1)(-a_i^- - b_i) + a_i^+ L_i - b_i \leq h_i \leq (y_i - 1)(a_i^+ - b_i) - a_i^- L_i - b_i$$

Case 2 (cont) : We now want to use Lemma 3.2 on $L_i \leq y_i - 1 \leq U_i - 1$

$$-t^-(U_i - 1) + t^+ L_i \leq t(y_i - 1) \leq t^+(U_i - 1) - t^- L_i$$

If we substitute the minima and maxima for $(y_i - 1)$ we get the following

$$-(a_i^- + b_i)^+(U_i - 1) + (a_i^- + b_i)^- L_i + a_i^+ L_i - b_i \leq h_i \leq (a_i^+ - b_i)^+(U_i - 1) - (a_i^+ - b_i)^- L_i - a_i^- L_i - b_i$$

$t = -(a^- + b)$ for the lower bound
$t = (a^+ - b)$ for the upper bound

Case 3 : D = “>”. $L_i \leq y_i < x_i \leq U_i$. Since x_i and y_i are integers we can do the following

$$L_i \leq y_i \leq x_i - 1 \leq U_i - 1$$
$$h_i = a_i x_i - b_i y_i = a_i(x_i - 1) - b_i y_i + a_i$$

Apply Lemma 3.2 to $L_i \leq y_i \leq x_i - 1$ and replace t with $-b_i$

$$-b_i^+(x_i - 1) + b_i^- L_i \leq b_i y_i \leq b_i^-(x_i - 1) - b_i^+ L_i$$

If we add $a_i + a_i(x_i - 1)$ to all sides we get

$$-b_i^+(x_i - 1) + b_i^- L_i + a_i + a_i(x_i - 1) \leq h_i \leq b_i^-(x_i - 1) - b_i^+ L_i + a_i + a_i(x_i - 1)$$

We can factor out $(x_i - 1)$

$$(x_i - 1)(-b_i^+ + a_i) + a_i - b_i^+ L_i \leq h_i \leq (x_i - 1)(b_i^- + a_i) + a_i - b_i^+ L_i$$

Case 3 (cont) : We now want to use Lemma 3.2 on $L_i \leq x_i - 1 \leq U_i - 1$

$$-t^-(U_i - 1) + t^+ L_i \leq t(x_i - 1) \leq t^+(U_i - 1) - t^- L_i$$

If we substitute the minima and maxima for $(x_i - 1)$ we get the following

$$-(a_i - b_i^+)^-(U_i - 1) + (a_i - b_i^+)^+ L_i + b_i^- L_i + a_i \leq h_i \leq (a_i + b_i^-)^+(U_i - 1) - (a_i + b_i^-)^- L_i - b_i^+ L_i + a_i$$

$t = (a_i - b_i^+)$ for the lower bound
$t = (a_i + b_i^-)$ for the upper bound

Case 4 : D = "*". $L_i^x \leq x_i \leq U_i^x$, $L_i^y \leq y_i \leq U_i^y$

Since $h_i = a_i x_i - b_i y_i$ we can use Lemma 3.2

$$-t^- U^x + t^+ L^x \leq t x_i \leq t^+ U^x - t^- L^x$$
$$-t^- U^y + t^+ L^y \leq t y_i \leq t^+ U^y - t^- L^y$$

Set $t = a_i$ in the first equation

Set $t = -b_i$ in the second equation

Summing the two equations proves our values for $H_i^+(*)$ and $H_i^-(*)$

Theorem 3.4 Banerjee's Inequality

There exists a real solution to $h = 0$ for direction vector D iff

$$\sum_{i=1}^{n} H_i^-(D_i) \leq b_0 - a_0 \leq \sum_{i=1}^{n} H_i^+(D_i) \qquad (1)$$

Proof of Theorem 3.4

Because of the dependence equation we can write

$$h = a_0 - b_0 + \sum_{i=1}^{n} h_i = a_0 - b_0 + \sum_{i=1}^{n} (a_i x_i - b_i y_i) \tag{2}$$

Using Lemma 3.3 we can minimize and maximize h

$$min_R h = a_0 - b_0 + \sum_{i=1}^{n} H_i^-(D_i) \tag{3}$$

$$max_R h = a_0 - b_0 + \sum_{i=1}^{n} H_i^+(D_i) \tag{4}$$

Substitute these values into Theorem 3.2 and subtract $(a_0 - b_0)$ from all sides

What about trapezoids?

```
DO I = 1,100
        DO J = 1,I-1
                A(J) = A(I+J-1) + C
        ENDDO
ENDDO
```

Primary insight

We want to consider bounds that are affine expressions

Trapezoidal Banerjee Inequality

The values V_{min} and V_{max} calculated in 3.8 are such that

$$\mathsf{V}_{min} \leq \mathsf{min}_R h \text{ and } \mathsf{max}_R h \leq \mathsf{V}_{max}$$

If this is false, then no dependencies can exist